

CHAPTER 3 THE RADIOACTIVE DECAY LAW

Introduction

We covered radioactive decay from a phenomenological perspective in the last chapter. In this chapter we consider the topic from a more general analytical perspective.

The reason for doing this is so that we can develop a form of thinking which will help us to understand what is going on in a quantitative, mathematical sense. We will be introduced to concepts such as the Decay Constant and the Half Life as well as units used for the measurement of radioactivity. You will also have a chance to develop your understanding by being brought through three questions on this subject.

Assumptions

The usual starting point in most forms of analysis in physics is to make some assumptions which simplify the situation. By simplifying the situation we can dispose of irrelevant effects which tend to complicate matters but in doing so we sometimes make the situation so simple that it becomes a bit too abstract and apparently hard to understand.

For this reason we will try here to relate the subject of radioactive decay to a more common situation which we will use as an analogy and hopefully we will be able to overcome the abstract feature of the subject matter. The analogy we will use here is that of making **popcorn**.

So think about putting some oil in a pot, adding the corn, heating the pot on the cooker and watching what happens. You might also like to try this out while considering the situation!

For our radioactive decay situation we first of all consider that we have a sample containing a large number of radioactive nuclei all of the same kind. This is our unpopped corn in the pot for example.

Secondly we assume that all of the radioactive nuclei decay by the same process be it alpha, beta or gamma-decay. In other words our unpopped corn goes pop at some stage during the heating process.

Thirdly take a few moments to ponder on the fact that we can only really consider what is going on from a statistical perspective. If you look at an individual piece of corn, can you figure out when it is going to pop? Not really. You can however figure out that a large number of them will have popped after a

period of time. But its rather more difficult to figure out the situation for an individual piece of corn. So instead of dealing with individual entities we consider what happens on a larger scale and this is where statistics comes in. We can say that the radioactive decay is a statistical one-shot process, that is when a nucleus has decayed it cannot repeat the process again. In other words when a piece of corn has popped it cannot repeat the process. Simple!

In addition as long as a radioactive nucleus has not decayed the probability for it doing so in the next moment remains the same. In other words if a piece of corn has not popped at a certain time the chance of it popping in the next second is the same as in the previous second. The bets are even!

Let us not push this popcorn analogy too far though in that we know that we can control the rate of popping by the heat we apply to the pot for example. However as far as our radioactive nuclei are concerned there is nothing we can do to control what is going on. The rate at which nuclei go pop (or decay, in other words) cannot be influenced by heating up the sample. Nor by cooling it for that matter or by putting it under greater pressures, by changing the gravitational environment by taking it out into space for instance, or by changing any other aspect of its physical environment. The only thing that determines whether an individual nucleus will decay seems to be the nucleus itself. But on the average we can say that it will decay at some stage.

The Radioactive Decay Law

Let us now use some symbols to reduce the amount of writing we have to do to describe what is going on and to avail ourselves of some mathematical techniques to simplify the situation even further than we have been able to do so far.

Let us say that in the sample of radioactive material there are N nuclei which have not decayed at a certain time, t . So what happens in the next brief period of time? Some nuclei will decay for sure. But how many?

On the basis of our reasoning above we can say that the number which will decay will depend on overall number of nuclei, N , and also on the length of the brief period of time. In other words the more nuclei there are the more will decay and the longer the time period the more nuclei will decay. Let us denote the number which will have decayed as dN and the small time interval as dt .

So we have reasoned that the number of radioactive nuclei which will decay during the time interval from t to $t+dt$ must be proportional to N and to dt . In symbols therefore:

$$-dN \propto N \cdot dt$$

the minus sign indicating that N is decreasing.

Turning the proportionality in this equation into an equality we can write:

$$-dN = \lambda N \cdot dt$$

where the constant of proportionality, λ , is called the Decay Constant.

Dividing across by N we can rewrite this equation as:

$$-\frac{dN}{N} = \lambda \cdot dt$$

So this equation describes the situation for any brief time interval, dt . To find out what happens for all periods of time we simply add up what happens in each brief time interval. In other words we integrate the above equation. Expressing this more formally we can say that for the period of time from $t = 0$ to any later time t , the number of radioactive nuclei will decrease from N_0 to N_t , so that:

$$-\int_{N_0}^{N_t} \frac{dN}{N} = \lambda \int_0^t dt$$

$$\therefore \ln \left(\frac{N_t}{N_0} \right) = -\lambda t$$

$$\therefore \frac{N_t}{N_0} = \exp (-\lambda t)$$

$$\therefore N_t = N_0 \exp (-\lambda t)$$

This final expression is known as the **Radioactive Decay Law**. It tells us that the number of radioactive nuclei will decrease in an exponential fashion with time with the rate of decrease being controlled by the Decay Constant.

Before looking at this expression in further detail let us review the mathematics which we used above. First of all we used integral calculus to figure out what was happening over a period of time by integrating what we knew would occur in a brief interval of time. Secondly we used a calculus relationship that the

$$\int \frac{dx}{x} = \ln x$$

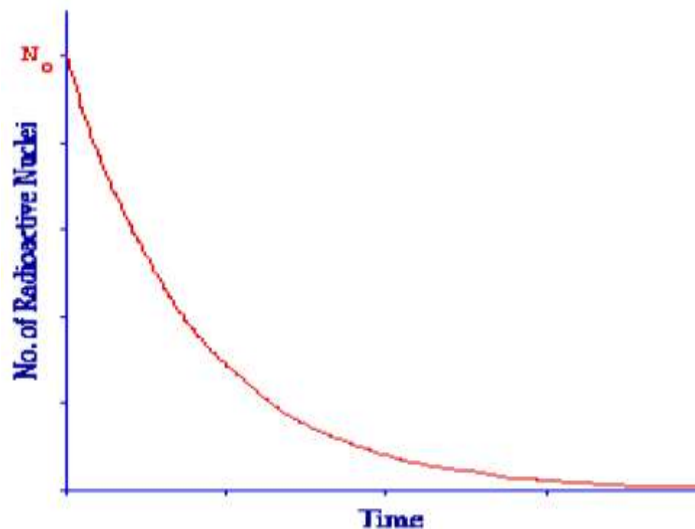
where $\ln x$ represents the natural logarithm of x . And thirdly we used the definition of logarithms that when

$$\ln x = y$$

then,

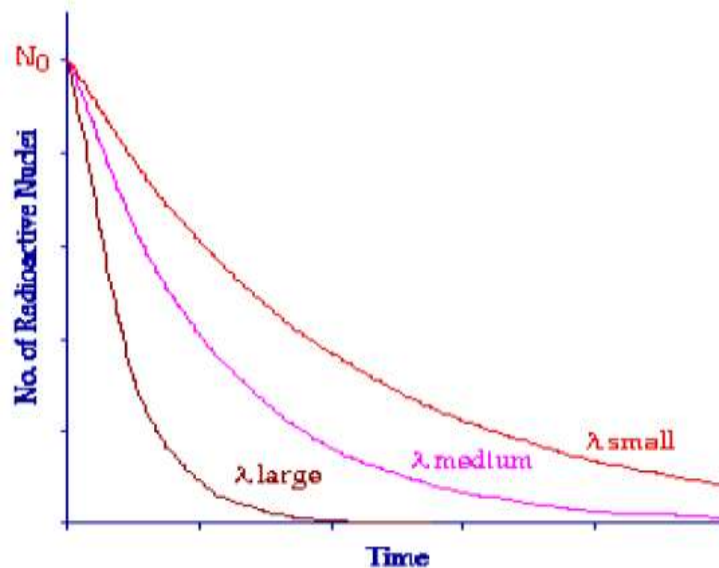
$$x = \exp y$$

Now, to return to the Radioactive Decay Law. The Law tells us that the number of radioactive nuclei will decrease with time in an exponential fashion with the rate of decrease being controlled by the Decay Constant. The Law is shown in graphical form in the figure below:



The graph plots the number of radioactive nuclei at any time, N_t , against time, t . We can see that the number of radioactive nuclei decreases from N_0 that is the number at $t = 0$ in a rapid fashion initially and then more slowly in the classic exponential manner.

The influence of the Decay Constant can be seen in the following figure:



All three curves here are exponential in nature, only the Decay Constant is different. Notice that when the Decay Constant has a low value the curve decreases relatively slowly and when the Decay Constant is large the curve decreases very quickly.

The Decay Constant is characteristic of individual radionuclides. Some like uranium-238 have a small value and the material therefore decays quite slowly over a long period of time. Other nuclei such as technetium-99m have a relatively large Decay Constant and they decay far more quickly.

It is also possible to consider the Radioactive Decay Law from another perspective by plotting the logarithm of N_t against time. In other words from our analysis above by plotting the expression:

$$\ln \left(\frac{N_t}{N_0} \right) = -\lambda t$$

in the form

$$\ln N_t = -\lambda t + \ln N_0$$

Notice that this expression is simply an equation of the form $y = mx + c$ where $m = -\lambda$ and $c = \ln N_0$. As a result it is the equation of a straight line of slope $-\lambda$ as shown in the following figure. Such a plot is sometimes useful when we wish to consider a situation without the complication of the direct exponential behaviour.



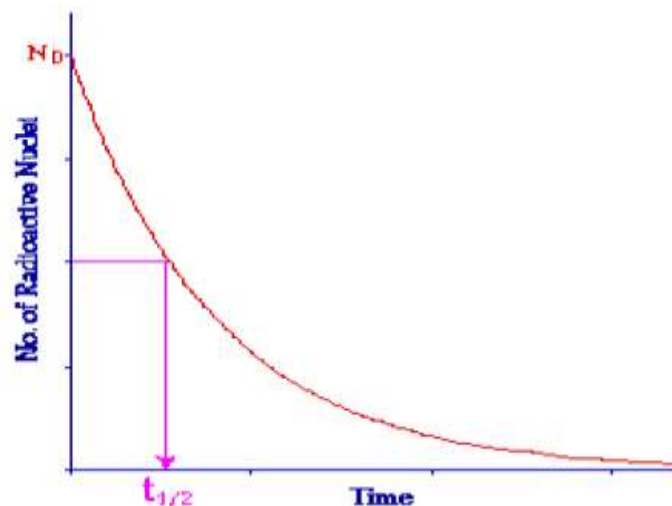
Half-Life

Most of us have not been taught to think instinctively in terms of logarithmic or exponential terms even though many natural phenomena display exponential behaviours. Most of the forms of thinking which we have been taught in school are based on linear changes and as a result it is rather difficult for us to grasp the Radioactive Decay Law intuitively. For this reason an indicator is usually derived from the law which helps us think more clearly about what is going on.

This indicator is called **the Half Life** and it expresses the length of time it takes for the radioactivity of a radioisotope to decrease by a factor of two. From a graphical point of view we can say that when:

$$N_t = \frac{N_0}{2}$$

the time taken is the Half Life:



Note that the half-life does not express how long a material will remain radioactive but simply the length of time for its radioactivity to halve. Examples of the half lives of some radioisotopes are given in the following table. Notice that some of these have a relatively short half life. These tend to be the ones used for medical diagnostic purposes because they do not remain radioactive for very long following administration to a patient and hence result in a relatively low radiation dose.

Radioisotope	Half Life (approx.)
$^{81\text{m}}\text{Kr}$	13 seconds
$^{99\text{m}}\text{Tc}$	6 hours
^{131}I	8 days
^{51}Cr	1 month
^{137}Cs	30 years
^{241}Am	462 years
^{226}Ra	1620 years
^{238}U	4.51×10^9 years

But they do present a logistical problem when we wish to use them when there may not be a radioisotope production facility nearby. For example, suppose we wish to use $^{99\text{m}}\text{Tc}$ for a patient study and the nearest nuclear facility for making this isotope is 5,000 km away. The production facility could be in Sydney and the patient could be in Perth for instance. After making the isotope at the nuclear plant it would be decaying with a half life of 6 hours. So we put the material on a truck and drive it to Sydney airport. The isotope would be decaying as the truck sits in Sydney traffic then decaying still more as it waits for a plane to take it to Perth. Then decaying more as it is flown across to Perth and so on. By the time it gets to our patient it will have substantially reduced in radioactivity possibly to the point of being useless for the patient's investigation. And what

about the problem if we needed to use ^{81}mKr instead of ^{99}mTc for our patient? You will see in the last chapter of this wikibook that logistical challenges such as this have given rise to quite innovative solutions. More about that later!

You can appreciate from the table above that other isotopes have a very long half lives. For example, ^{226}Ra has a half life of over 1,500 years. This isotope has been used in the past for therapeutic applications in medicine. Think about the logistical problems here. They obviously do not relate to transporting the material from the point of production to the point of use. But they relate to how the material is kept following its arrival at the point of use. We must have a storage facility so that the material can be kept safely for a long period of time. But for how long? A general rule of thumb for the quantities of radioactivity used in medicine is that the radioactivity will remain significant for about 10 half lives. So we would have to have a safe environment for storage of the ^{226}Ra for about 16,000 years! This storage facility would have to be secure from many unforeseeable events such as earthquakes, bombing etc and be kept in a manner which our children's, children's children can understand. A very serious undertaking indeed!

Relationship between the Decay Constant and the Half Life

On the basis of the above you should be able to appreciate that there is a relationship between the Decay Constant and the Half Life. For example, when the Decay Constant is small the Half Life should be long and correspondingly when the Decay Constant is large the Half Life should be short. But what exactly is the nature of this relationship?

We can easily answer this question by using the definition of Half Life and applying it to the Radioactive Decay Law.

Once again the law tells us that at any time, t :

$$N_t = N_0 \exp(-\lambda t)$$

and the definition of Half Life tells us that:

$$N_t = \frac{N_0}{2}$$

when

$$t = t_{\frac{1}{2}}$$

We can therefore re-write the Radioactive Decay Law by substituting for N_t and t as follows:

Therefore

$$\frac{N_0}{2} = N_0 \exp(-\lambda t_{\frac{1}{2}})$$

$$\therefore 2^{-1} = \exp(-\lambda t_{\frac{1}{2}})$$

$$\therefore \ln 2^{-1} = -\lambda t_{\frac{1}{2}}$$

$$\therefore \ln 2 = \lambda t_{\frac{1}{2}}$$

$$\therefore 0.693 = \lambda t_{\frac{1}{2}}$$

and

$$t_{\frac{1}{2}} = \frac{0.693}{\lambda}$$

$$\lambda = \frac{0.693}{t_{\frac{1}{2}}}$$

These last two equations express the relationship between the Decay Constant and the Half Life. They are very useful as you will see when solving numerical questions relating to radioactivity and usually form the first step in solving a numerical problem.

Units of Radioactivity

The SI or metric unit of radioactivity is named after Henri Becquerel, in honour of his discovery of radioactivity, and is called the becquerel with the symbol Bq. The becquerel is defined as the quantity of radioactive substance that gives rise to a decay rate of 1 decay per second.

In medical diagnostic work 1 Bq is a rather small amount of radioactivity. Indeed it is easy to remember its definition if you think of it as a buggerall amount of

radioactivity. For this reason the kilobecquerel (kBq) and megabecquerel (MBq) are more frequently used.

The traditional unit of radioactivity is named after Marie Curie and is called the curie, with the symbol Ci. The curie is defined as the amount of radioactive substance which gives rise to a decay rate of 3.7×10^{10} decays per second. In other words 37 thousand, million decays per second which as you might appreciate is a substantial amount of radioactivity. For medical diagnostic work the millicurie (mCi) and the microcurie (μ Ci) are therefore more frequently used.

Why two units? It in essence like all other units of measurement depends on what part of the world you are in. For example the kilometer is widely used in Europe and Australia as a unit of distance and the mile is used in the USA. So if you are reading an American textbook you are likely to find the curie used as the unit of radioactivity, if you are reading an Australian book it will most likely refer to becquerels and both units might be used if you are reading a European book. You will therefore find it necessary to know and understand both units.

Questions

Three questions are given below to help you develop your understanding of the material presented in this chapter. The first one is relatively straight-forward and will exercise your application of the Radioactive Decay Law as well as your understanding of the concept of Half Life. The second question is a lot more challenging and will help you relate the Radioactive Decay Law to the number of radioactive nuclei which are decaying in a sample of radioactive material. The third question will help you understand the approach used in the second question by asking a similar question from a slightly different perspective.

Question 1

- (a) The half-life of ^{99m}Tc is 6 hours. After how much time will 1/16th of the radioisotope remain?
- (b) Verify your answer by another means.

Answer:

(a) Starting with the relationship we established earlier between the Decay Constant and the Half Life we can calculate the Decay Constant as follows:

$$\lambda = \frac{0.693}{t_{\frac{1}{2}}} = \frac{0.693}{6} = 0.1155 \text{ hr}^{-1}$$

Now applying the Radioactive Decay Law,

$$N_t = N_0 \exp (-\lambda t)$$

we can re-write it in the form:

The question tells us that N_0 has reduced to 1/16th of its value, that is:

$$\text{Therefore } \frac{N_t}{N_0} = \exp (-\lambda t)$$

$$\frac{1}{16} = \exp (-0.1155t)$$

which we need to solve for t . One way of doing this is as follows:

$$16^{-1} = \exp (-0.1155t)$$

$$t = \frac{-\ln 16}{0.1155} = 24 \text{ hours}$$

So it will take 24 hours until 1/16th of the radioactivity remains.

(b) A way in which this answer can be verified is by using the definition of Half Life. We are told that the Half Life of ^{99m}Tc is 6 hours. Therefore, after six hours half of the radioactivity remains.

Therefore, after 12 hours a quarter remains; after 18 hours an eighth remains and after 24 hours one sixteenth remains. And we arrive at the same answer as in part (a). So we must be right!

Note that this second approach is useful if we are dealing with relatively simple situations where the radioactivity is halved, quartered and so on. But supposing the question asked how long would it take for the radioactivity to decrease to a tenth of

its initial value. Deduction from the definition of half-life is rather more difficult in this case and the mathematical approach used for part (a) above will yield the answer more readily.

Question 2

Find the radioactivity of a 1 g sample of ^{226}Ra given that $t_{1/2}$: 1620 years and Avogadro's Number: 6.023×10^{23} .

Answer:

We can start the answer like we did with Question 1(a) by calculating the Decay Constant from the Half Life using the following equation:

$$\lambda = \frac{0.693}{t_{1/2}} = \frac{0.693}{1620} = 4.28 \times 10^{-4} \text{ year}^{-1}$$

$$\therefore \lambda = 1.36 \times 10^{-11} \text{ s}^{-1}$$

Note that the length of a year used in converting from 'per year' to 'per second' above is 365.25 days to account for leap years. In addition, the reason for converting to units of 'per second' is because the unit of radioactivity is expressed as the number of nuclei decaying per second.

Secondly we can calculate that 1 g of ^{226}Ra contains:

$$N = \frac{(\text{Avogadro's No.})(\text{Mass})}{\text{Mass Number}} = \frac{(6.023 \times 10^{23})(1\text{g})}{226} = 2.7 \times 10^{21} \text{ nuclei}$$

Thirdly we need to express the Radioactive Decay Law in terms of the number of nuclei decaying per unit time. We can do this by **differentiating** the equation as follows:

$$N = N_0 \exp(-\lambda t)$$

$$\therefore \frac{dN}{dt} = N_0 \cdot -\lambda \exp(-\lambda t) = -\lambda N_0 \exp(-\lambda t)$$

$$\therefore \frac{dN}{dt} = -\lambda N$$

$$\therefore \left| \frac{dN}{dt} \right| = \lambda N$$

The reason for expressing the result above in absolute terms is to remove the minus sign in that we already know that the number is decreasing.

We can now enter the data we derived above for λ and N :

$$\left| \frac{dN}{dt} \right| = (1.36 \times 10^{-11})(2.7 \times 10^{21})$$

$$\therefore \left| \frac{dN}{dt} \right| = 3.6 \times 10^{10} \text{ decays per second}$$

So the radioactivity of our 1 g sample of radium-226 is approximately 1 Ci.

This is not a surprising answer since the definition of the **curie** was originally conceived as the radioactivity of 1 g of radium-226!

Question 3

What is the minimum mass of ^{99m}Tc that can have a radioactivity of 1 MBq? Assume the half-life is 6 hours and that Avogadro's Number is 6.023×10^{23} .

Answer

Starting again with the relationship between the Decay Constant and the Half Life:

$$\lambda = \frac{0.693}{6} = 0.1155 \text{ hour}^{-1} = 3.21 \times 10^{-5} \text{ s}^{-1}$$

Secondly the question tells us that the radioactivity is 1 MBq. Therefore since $1 \text{ MBq} = 1 \times 10^6 \text{ decays per second}$,

$$\left| \frac{dN}{dt} \right| = \lambda N = 1 \times 10^6 \text{ dps}$$

Finally the mass of these nuclei can be calculated as follows:

$$\text{Mass of } N \text{ nuclei} = \frac{(\text{No. of Nuclei})(\text{Mass No.})}{\text{Avogadro's Number}}$$

$$= \frac{(3.116 \times 10^{10})(99)}{6.023 \times 10^{23}} = 5.122 \times 10^{-12} \text{ g}$$

In other words a mass of just over five picograms of ^{99m}Tc can emit one million gamma-rays per second. The result reinforces an important point that you will learn about radiation protection which is that you should treat radioactive materials just like you would handle pathogenic bacteria!